Free choosability of outerplanar graphs

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FREE CHOOSABILITY OF THE CYCLE

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Abstract. A graph G is free (a, b)-choosable if for any vertex v with b colors assigned and for any list of colors of size a associated with each vertex $u \neq v$, the coloring can be completed by choosing for u a subset of b colors such that adjacent vertices are colored with disjoint color sets. In this note, a necessary and sufficient condition for a cycle to be free (a, b)-choosable is given. As a corollary, some choosability results are derived for graphs in which cycles are connected by a tree structure.

1. Introduction

For a graph G, we denote its vertex set by $V(G)$ and edge set by $E(G)$. A color-list L of a graph G is an assignment of lists of integers (colors) to the vertices of G. For an integer a, a a-color-list L of G is a color-list such that $|L(v)| = a$ for any $v \in V(G)$.

In 1996, Voigt considered the following problem: let G be a graph and L a color-list and assume that an arbitrary vertex $v \in V(G)$ is precolored by a color $f \in L(v)$. Under which hypothesis is it always possible to complete this precoloring to a proper color-list coloring ? This question leads to the concept of free choosability introduced by Voigt [8].

Formally, for a graph G, integers a, b and a a-color-list L of G, an (L, b)-*coloring* of G is a mapping c that associates to each vertex u a subset $c(u)$ of $L(u)$ such that $|c(u)| = b$ and $c(u) \cap c(v) = \emptyset$ for any edge $uv \in E(G)$. The graph G is (a, b)-*choosable* if for any a-color-list L of G, there exists an (L, b)-coloring . Moreover, G is *free* (a, b)-*choosable* if for any a-color-list L, any vertex v and any set $c_0 \subset L(v)$ of b colors, there exists an (L, b)-coloring c such that $c(v) = c_0$.

As shown by Voigt [8], there are examples of graphs G that are (a, b)-choosable but not free (a, b)-choosable. Some related recent results concern defective free choosability of planar graphs [6]. We investigate, in the next section, the free-choosability of the first interesting case, namely the cycle. We derive a necessary and sufficient condition for a cycle to be free (a, b)-choosable (Theorem 4). In order to get a concise statement, we introduce the free-choice ratio of a graph, in the same way that Alon, Tuza and Voigt [1] introduced the choice ratio (which is equal to the so-called fractional chromatic number).

For any real x, let FCH(x) be the set of graphs G which are free (a, b)-choosable for all a, b such that $\frac{a}{b} \geq x$:

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$$\mathrm{FCH}(x) = \Big\{ G \mid \forall\ \frac{a}{b} \geq x,\ G \text{ is free } (a,b)\text{-choosable} \Big\}.$$

Moreover, we can define the free-choice ratio $\mathrm{fchr}(G)$ of a graph G by:

$$\mathrm{fchr}(G) := \inf \Big\{ \frac{a}{b} \mid G \text{ is free } (a,b)\text{-choosable} \Big\}.$$

Remark 1. *Erdős, Rubin and Taylor have asked* [3] *the following question: If G is (a,b)-choosable, and $\frac{c}{d} > \frac{a}{b}$, does it imply that G is (c,d)-choosable ? Gutner and Tarsi have shown* [5] *that the answer is negative in general. If we consider the analogue question for the free choosability, then Theorem 4 implies that it is true for the cycle.*

The *path* P_{n+1} of length n is the graph with vertex set $V = \{v_0, v_1, \dots, v_n\}$ and edge set $E = \bigcup_{i=0}^{n-1}\{v_i v_{i+1}\}$. The cycle C_n of length n is the graph with vertex set $V = \{v_0, \dots, v_{n-1}\}$ and edge set $E = \bigcup_{i=0}^{n-1}\{v_i v_{i+1(mod\ n)}\}$. To simplify the notation, for a color-list L of P_n or C_n, we let $L(i)$ denote $L(v_i)$ and $c(i)$ denote $c(v_i)$.

The notion of waterfall color-list was introduced in [2] to obtain choosability results on the weighted path and then used to prove the $(5m, 2m)$-choosability of triangle-free induced subgraphs of the triangular lattice. We recall one of the results from [2] that will be used in this note, with the function Even being defined for any real x by: $\mathrm{Even}(x)$ is the smallest even integer p such that $p \geq x$.

Proposition 2 ([2]). *Let L be a color-list of P_{n+1} such that $|L(0)| = |L(n)| = b$, and $|L(i)| = a = 2b + e$ for all $i \in \{1, \dots, n-1\}$ (with $e > 0$).*

$$\text{If } n \geq \mathrm{Even}\Big(\tfrac{2b}{e}\Big) \text{ then } P_{n+1} \text{ is } (L,b)\text{-colorable.}$$

For example, let P_{n+1} be the path of length n with a color-list L such that $|L(0)| = |L(n)| = 4$, and $|L(i)| = 9$ for all $i \in \{1, \dots, n-1\}$. Then the previous proposition tells us that we can find an $(L,4)$-coloring of P_{n+1} whenever $n \geq 8$. In other words, if $n \geq 8$, we can choose 4 colors on each vertex such that adjacent vertices receive disjoint colors sets. If $|L(i)| = 11$ for all $i \in \{1, \dots, n-1\}$, then P_{n+1} is $(L,4)$-colorable whenever $n \geq 4$. On the other side, there are color-lists L for which P_{n+1} is not (L,b)-colorable

2. Free choosability of the cycle

We begin with a negative result for the even-length cycle:

Lemma 3. *For any integers a, b, p such that $p \geq 2$, and $\frac{a}{b} < 2 + \frac{1}{p}$, the cycle C_{2p} is not free (a,b)-choosable.*

Proof. We construct a counterexample for the free-choosability of C_{2p}: let L be the a-color-list of C_{2p} such that

$$L(i) = \begin{cases} \{1, \dots, a\}, & \text{if } i \in \{0,1\}; \\ \{\frac{i-1}{2}a + 1, \dots, \frac{i-1}{2}a + a\}, & \text{if } i \neq 2p-1 \text{ is odd;} \\ \{b + \frac{i-2}{2}a + 1, \dots, b + (\frac{i-2}{2} + 1)a\}, & \text{if } i \text{ is even and } i \neq 0; \\ \{1, \dots, b, 1 + (p-1)a, \dots, 1 + pa - b - 1\}, & \text{if } i = 2p-1. \end{cases}$$

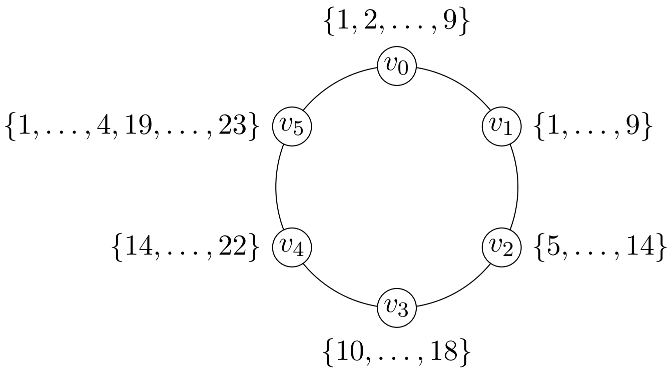

FIGURE 1. The cycle C_6, along with a 9-color-list L for which there is no $(L, 4)$-coloring c such that $c(v_0) = \{1, 2, 3, 4\}$.

If we choose $c_0 = \{1, \ldots, b\} \subset L(0)$, we can check that there does not exist an (L, b)-coloring of C_{2p} such that $c(0) = c_0$, so C_{2p} is not free (a, b)-choosable. See Figure 1 for an illustration when $p = 3$, $a = 9$ and $b = 4$. $\square$

Now, if $\lfloor x \rfloor$ denotes the greatest integer less than or equal to the real x, we can state:

Theorem 4. *For the cycle C_n of length n,*

$$C_n \in \mathrm{FCH}\left(2 + \left\lfloor \frac{n}{2} \right\rfloor^{-1}\right).$$

Moreover, we have:

$$\mathrm{fchr}(C_n) = 2 + \left\lfloor \frac{n}{2} \right\rfloor^{-1}.$$

Proof. Let a, b be two integers such that $a/b \geq 2 + \lfloor \frac{n}{2} \rfloor^{-1}$. Let L be a a-color-list of C_n. Without loss of generality, we can suppose that v_0 is the vertex chosen for the free-choosability and $c_0 \subset L(v_0)$ has b elements. It remains to construct an (L, b)-coloring c of C_n such that $c(v_0) = c_0$. Hence we have to construct an (L', b)-coloring c of P_{n+1} such that $L'(0) = L'(n) = L_0$ and for all $i \in \{1, ..., n-1\}$, $L'(i) = L(v_i)$. We have $|L'(0)| = |L'(n)| = b$ and for all $i \in \{1, ..., n-1\}$, $|L'(i)| = a$. Since $a/b \geq 2 + \lfloor \frac{n}{2} \rfloor^{-1}$ and $e = a - 2b$, we get $e/b \geq \lfloor \frac{n}{2} \rfloor^{-1}$ hence $n \geq \mathrm{Even}(2b/e)$. Using Proposition 2, we get:

$$C_n \in \mathrm{FCH}(2 + \left\lfloor \frac{n}{2} \right\rfloor^{-1}).$$

Hence, we have that $\mathrm{fchr}(C_n) \leq 2 + \lfloor \frac{n}{2} \rfloor^{-1}$. Moreover, let us prove that $M = 2 + \lfloor \frac{n}{2} \rfloor^{-1}$ is reached. For n odd, Voigt has proved [9] that the choice ratio $\mathrm{chr}(C_n)$ of a cycle of odd length n is exactly M. Hence $\mathrm{fchr}(C_n) \geq \mathrm{chr}(C_n) = M$, and the result is proved. For n even, Lemma 3 asserts that C_n is not free (a, b)-choosable for $\frac{a}{b} < 2 + \lfloor \frac{n}{2} \rfloor^{-1}$. $\square$

Remark 5. *In particular, the previous theorem implies that if b, e, n are integers such that $n \geq \mathrm{Even}(\frac{2b}{e})$, then the cycle C_n of length n is free $(2b + e, b)$-choosable.*

In order to extend the result to other graphs then cycles, the following simple proposition will be useful:

Proposition 6. *Let a, b be integers with $a \geq 2b$. Let G be a graph and G_v be the graph obtained by adding a leave v to any vertex of G. Then G is free (a, b)-choosable if and only if G_v is free (a, b)-choosable.*

Proof. Since the "only if" part holds trivially, let us prove the "if" part. Assume G is free (a, b)-choosable and let L be a a-color-list of G. Let v be a new vertex and let G_v be the graph obtained from G by adding the edge uv, for some $u \in V(G)$. Then any (L, b)-coloring c of G can be extended to an (L, b)-coloring of G_v by giving to v b colors from $L(v) \setminus c(u)$ ($|L(v) \setminus c(u)| \geq b$ since $a \geq 2b$). If v is colored with b colors from its list, then, since G is free (a, b)-choosable, the coloring can be extended to an (L, b)-coloring of G_v by first choosing for u a set of b colors from $L(u) \setminus c(v)$. □

Starting from a single edge and applying inductively Proposition 6 allows to obtain the following corollary:

Corollary 7. *Let T be a tree of order $n \geq 2$. Then*

$$T \in \mathrm{FCH}(2).$$

Now, we can state the following:

Proposition 8. *If G is a unicyclic graph with girth g, then*

$$G \in \mathrm{FCH}\left(2 + \left\lfloor \frac{g}{2} \right\rfloor^{-1}\right).$$

Proof. Let a, b be two integers such that $a/b \geq 2 + \lfloor \frac{n}{2} \rfloor^{-1}$, L be a a-color-list of G, $C = v_1, \ldots, v_g$ be the unique cycle (of length g) of G and T_i, $i \in \{1, \ldots, g\}$, be the subtree of G rooted at vertex v_i of C.

Let v be the vertex chosen for the free choosability and let $c_0 \subset L(v)$ be a set of cardinality b. If $v \in C$, then by virtue of Theorem 4, there exists an (L, b)-coloring c of C such that $c(v) = c_0$. This coloring can be easily extended to the whole graph by coloring the vertices of each tree T_i thanks to Corollary 7. If $v \in T_i$ for some i, $1 \leq i \leq g$, then Corollary 7 asserts that the coloring can be extended to T_i. Then color C starting at vertex v_i by using Theorem 4. Finally, complete it by coloring each tree T_j, $1 \leq j \neq i \leq g$. □

3. Applications

As an example to the possible use of the results from Section 2, we begin with determining the free choosability of a binocular graph, i.e. two cycles linked by a path.

For integers m, n and p such that $m, n \geq 3$ and $p \geq 0$, the *binocular graph* $\mathrm{BG}(m, n, p)$ is the disjoint union of an m-cycle $u_0, u_1, \ldots, u_{m-1}$ and of an n-cycle $v_0, \ldots, v_{n-1}$ with vertices u_0 and v_0 linked by a path of length p given by $u_0, x_1, \ldots, x_{p-1}, v_0$. Note that if $p = 0$, then u_0 and v_0 are the same vertex.

Proposition 9. *For any $m \geq 3$, $n \geq 3$ and $p \geq 0$,*

$$BG(m, n, p) \in \mathrm{FCH}\left(2 + \left\lfloor \frac{\min(m, n)}{2} \right\rfloor^{-1}\right).$$

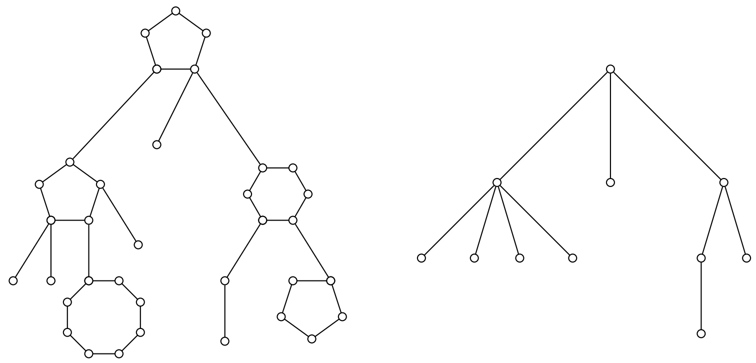

FIGURE 2. A tree of cycles (on the left), and the associated tree obtained by collapsing cycles (on the right).

Proof. Assume without loss of generality that $m \geq n$ and let a, b be integers such that $a/b \geq 2 + \lfloor \frac{n}{2} \rfloor^{-1}$. Let L be a a-color-list of $\mathrm{BG}(m, n, p)$. Let y be the vertex chosen for the free choosability and let $c_0 \subset L(y)$ be a set of cardinality b. If y lies on the m-cycle, then by virtue of Theorem 4, there exists an (L, b)-coloring c of the m-cycle such that $c(y) = c_0$. By Corollary 7, this coloring can be extended to the vertices of the path. Now, it remains to color the vertices of the n-cycle, with v_0 being already colored. This can be done thanks to Theorem 4. If y lies on the n-cycle, the argument is similar. If $y \in \{x_1, \ldots, x_{p-1}\}$, then the coloring can be extended to the whole path and the coloring of the m-cycle and n-cycle can be completed thanks to Theorem 4. □

This method can be extended to prove similar results on graphs with more than two cycles, connected by a tree structure.

Define a *tree of cycles* to be a graph G such that all its cycles are disjoint and collapsing all vertices of each cycle of G produces a tree.

Corollary 10. *Any tree of cycles of girth g is in* $\mathrm{FCH}(2 + \lfloor \frac{g}{2} \rfloor^{-1})$.

4. Algorithmic considerations

Let $n \geq 3$ be an integer and let a, b be two integers such that $a/b \geq 2 + \lfloor \frac{n}{2} \rfloor^{-1}$. Let L be a a-color-list of C_n.

As defined in [2], a *waterfall* list L of a path P_{n+1} of length n is a list L such that for all $i, j \in \{0, \ldots, n\}$ with $|i - j| \geq 2$, we have $L(i) \cap L(j) = \emptyset$. Le $m = | \cup_{i=0}^{n} L(i)|$ be the total number of colors of the color-list L.

The algorithm behind the proof of Proposition 2 consists in three steps: first, the transformation of the list L into a waterfall list L' by renaming some colors; second, the construction of the (L', b)-coloring by coloring vertices from 0 to $n - 1$, giving to vertex i the first b-colors that are not used by the previous vertex; third, the backward transformation to obtain an (L, b)-coloring from the (L', b)-coloring by coming back to original colors and resolving color conflicts if any. It can be seen that the time complexity of the first step is $O(mn)$; that of the second one is $O(a^2 n)$ and that of the third one is $O(\max(a, b^3)n)$. Therefore, the total running time for computing a free (L, b)-coloring of the cycle C_n is $O(\max(m, a^2, b^3)n)$.

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